Introduction to Symbolic Logic

David W. Agler
YouTube: www.youtube.com/c/LogicPhilosophy/
Twitter: twitter.com/davidagler
Website: www.davidagler.com
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Two notions of entailment
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• Intuitively, a good argument is one where the conclusion follows from (entails) the premises or assumptions.
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A set of wffs can be said to entail a wff in two different ways.
Two notions of entailment

1. A wff is **semantically entailed** by a set wff iff there is no way to interpret the premises and the conclusion such that the premises are true and the conclusion is false.
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2. A wff is **syntactically entailed** by a set of wffs iff there is a proof (or derivation) of that wff.
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2. A wff is **syntactically entailed** by a set of wffs iff there is a proof (or derivation) of that wff.
   - This is the notion of syntactic entailment or proof-theoretic entailment.
Derivations, entailment, the apparatus
To get a clearer sense of this second notion of entailment, we need to clarify a couple notions:
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2. **deductive apparatus**: the various rules that allow for taking steps in a proof.
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1. the notion of a **derivation** (or proof),
2. **deductive apparatus**: the various rules that allow for taking steps in a proof.

Once we clarify these notions, we will give a precise definition of when a wff is syntactically entailed by a set of wffs.
what is a derivation (proof)?

Definition (derivation of Q in PD)

A derivation (proof) in PD of Q is a finite (not infinite and not empty) string of wffs from a set Γ of PL wffs where (i) the last formula in the string is Q and (ii) each formula is either a premise, an assumption, or is the result of the preceding formulas and the deductive apparatus.
what is a derivation (proof)?

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• just a string of wffs that meets certain conditions.
• Last wff in the string needs to be the wff that is proved (aka the conclusion)
• Every wff needs to either be a premise, a wff that is assumed, or a wff that is proved using the deductive apparatus.
What is a deductive apparatus?

**Definition (deductive apparatus)**

A deductive apparatus for PL is a set of rules of inference (or “derivation” rules) that determine which ways that formulas can be transformed. That is, it is a list of permissible rules that express which wffs Q can be written after which formulas P in a proof. The deductive apparatus for PL is hereafter abbreviated as PD.
what is a deductive apparatus?

We can think of the deductive apparatus as motivated by arguing in everyday life.

- Suppose two people Tek and Liz agree on a great many things. They have similar life experiences, they read many of the same scientific studies, and they have similar values. Let’s refer to the set of propositions that Tek and Liz both take as true $\Gamma$ (where $\Gamma$ just represents a set of propositions, e.g. $A, B, C, \ldots M$).

- Now suppose that Tek thinks that from $\Gamma$, we can easily reason to another set of propositions $P, Q, R$. Tek contends that if we believe $\Gamma$ then we ought to also believe $P, Q, R$.

- In contrast, Liz says that even if all of the propositions in $\Gamma$ are true, there is no way to reason to $P, Q, R$. 
what is a deductive apparatus?

- Tek and Liz don’t disagree about any facts concerning the world.
- They disagree about is whether $P, Q, R$ follows from $\Gamma$.
- To fix this problem, they decide to develop a set of rules that specify how one can reason from one proposition (or groups of propositions) to another. The rules are formulated in a highly general (abstract) way so that it can apply to any particular subject matter. This set of rules is their deductive apparatus.

What, or what follows from what.
Definition (syntactic consequence (entailment))

A formula $Q$ is a syntactic consequence in $PD$ of a set $\Gamma$ of $PL$ wffs if and only if there is a derivation in $PD$ of $Q$ from $\Gamma$. 
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To express that $Q$ is a syntactic consequence of $\Gamma$, we write $\Gamma \vdash Q$. 
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To express that $Q$ is a syntactic consequence of $\Gamma$, we write $\Gamma \vdash Q$.

- $P, M, R \vdash Q$
- What is said here is that $Q$ is a syntactic consequence (is syntactically entailed by $P, M, R$
- In other words, there is a proof that begins with $P, M, R$ and ends with $Q$. 
How to set up a proof (Fitch-Style)
Now that we know what a proof is, we now will look at how to set up a proof.

• Consider the following: \( P \land R, Y \rightarrow R \vdash Z \).
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**Question**

How do we show that there is such a derivation?
Derivation Setup (Fitch-Style)

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First, a derivation begins with an initial setup involving three columns:

1. for numbering the premises,
2. writing (stacking) the propositions,
3. justification of propositions and indicating the goal proposition (or conclusion)
In the setup of the above derivation, $P \land R$ and $Y \rightarrow R$ are premises (and we use $P$) to indicate this. The conclusion (goal proposition) is $Z$. 
Intelim Derivation Rules
The Deductive Apparatus

- defined what a syntactic entailment is
The Deductive Apparatus

- defined what a syntactic entailment is
- defined what a proof is

What we don't have is the deductive apparatus itself!
The Deductive Apparatus

- defined what a syntactic entailment is
- defined what a proof is
- defined what a deductive apparatus is
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The Deductive Apparatus

- Remember the deductive apparatus is just a set of rules that allow us to write new wffs in the proof.
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- Remember the deductive apparatus is just a set of rules that allow us to write new wffs in the proof.
- In other words, they are rules that allow us to go from the premises to the conclusion.
The particular type of deductive apparatus developed here is known as a system of “natural deduction” as the particular rules are akin to certain rules of inference (or reason) people use in everyday arguments.
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The particular rules of PD will be called the “derivation rules”.
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2. **elimination rules** (these begin from propositions of a certain type in a proof).
**Conjunction Introduction**

From $P$ and $Q$, we can derive $P \land Q$. Also, from $P$ and $Q$, we can derive $Q \land P$. 

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Conjunction Introduction \( \land \)

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\[
P, Q \vdash P \land Q
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1. “John is a man”
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1. “John is a man”
2. “John is a banker”
PD Conjunction Introduction

The idea here is we can reason from two separate wffs to their conjunction.

1. “John is a man”
2. “John is a banker”
3. *Therefore*, “John is a man and a banker.”
PD Conjunction Introduction

1  \( P \)  P
2  \( Q \)  P
3  \( P \land Q \)  \( \land I, 1, 2 \)
Conjunction introduction states that from two different propositions, we can derive the conjunction of these propositions. For example, prove:

\[ P \rightarrow Q, R \lor M, Z \vdash (P \rightarrow Q) \land (R \lor M) \]
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<td>((P \rightarrow Q) \land (R \lor M))</td>
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Conjunction Elimination ($\land E$)

From $P \land Q$, we can derive $P$. Also, from $P \land Q$, we can derive $Q$. 
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From $P \land Q$, we can derive $P$. Also, from $P \land Q$, we can derive $Q$.

$P \land Q \vdash P$ or $P \land Q \vdash Q$
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2. “Therefore, Sally is a doctor”
3. “Therefore, Sally is a woman”
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\[
\begin{array}{c}
1 & P \land Z & P \\
\end{array}
\]
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\[
\begin{array}{c|l}
1 & P \land Z & P \\
2 & Z & \land E, 1 \\
\end{array}
\]
PD Conjunction Elimination

\[(P \land Z) \land Q \vdash Z\]
PD Conjunction Elimination

\[(P \land Z) \land Q \vdash Z\]

1 \hspace{1cm} (P \land Z) \land Q \quad P
PD Conjunction Elimination

\[(P \land Z) \land Q \vdash Z\]

1. \((P \land Z) \land Q\) \hspace{1cm} \text{P}
2. \(P \land Z\) \hspace{1cm} \land E, 1
PD Conjunction Elimination

\[(P \land Z) \land Q \vdash Z\]

1. \((P \land Z) \land Q\) \hspace{1cm} P
2. \(P \land Z\) \hspace{1cm} \land E, 1
3. \(Z\) \hspace{1cm} \land E, 2
An assumption (abbreviated as A) is a wff taken to be, or assumed, true for the purpose of proof. To indicate that an assumption is being made in the proof:
Assumptions and Subproofs

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2. draw a vertical line indicating the beginning of a subproof (a proof that is under an assumption)
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1. indent from the main part of the proof
2. draw a vertical line indicating the beginning of a **subproof** (a proof that is under an assumption)
3. and justify that proposition you assumed with an A.
## Assumptions and Subproofs

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<td>2</td>
<td>Assumption</td>
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<td>3</td>
<td>Subproof</td>
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<td>4</td>
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<td>5</td>
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Some quick notes about assumptions
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Assumptions and Subproofs

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- Making an assumption and reasoning under that assumption is something everyone does.
  1. Suppose you and a friend are having an argument.
  2. You might say to your friend, “let's assume what you are saying is true” and then reason based on that assumption.
Assumptions and Subproofs

After you have made an assumption, you can reason within the subproof that has been created by the assumption:
After you have made an assumption, you can reason within the subproof that has been created by the assumption:

1. \( S \)
2. \( B \)
3. \( S \land B \)
4.  
5.  
6.  

The above is similar to saying let's agree that \( S \) is true. Now, let's assume \( B \) is true. Well, if \( S \) is true, then given our assumption \( B \), it follows that \( S \land B \).
Assumptions and Subproofs

After you have made an assumption, you can reason within the subproof that has been created by the assumption:

\[
\begin{array}{c|c|c}
1 & S & P \\
2 & B & A \\
3 & S \land B & \land I, 1, 2 \\
4 & \text{.} & \\
5 & \text{.} & \\
6 & \text{.} & \\
\end{array}
\]

The above is similar to saying Let's agree that $S$ is true. Now, let's assume $B$ is true. Well, if $S$ is true, then given our assumption $B$, it follows that $S \land B$. 


You are not limited to one assumption. You can make assumptions within assumptions. For example, consider the proof just below.
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1. \( Q \) \hspace{1cm} P
2. \( S \) \hspace{1cm} A
3. \( W \) \hspace{1cm} A
4. 
5. 
6. 
The prior proof reads something like the following: *Let's say that Q is true. Given that Q is true, let's assume S. Now that we've assumed S, let's assume W.*
Assumptions and Subproofs

You can think of subproofs like containers or nests. That is the subproof begun by S contains the subproof begun by W. Likewise, the mainline of the proof, beginning with Q contains the subproofs begun by S and W. In the language of nests, W is in the nest begun by S and S is in the nest of the main line of the proof. W is in the most deeply nested part of the proof while Q is in the least deeply nested part.
Assumptions and subproofs can be independent of each other.
Assumptions and subproofs can be independent of each other.

1. \( A \) \hspace{1cm} \text{P}
2. \( B \) \hspace{1cm} \text{A}
3. \( A \wedge B \) \hspace{1cm} \( \wedge I, 1, 2 \)
4. .
5. .
6. \( C \) \hspace{1cm} \text{A}
7. \( A \wedge C \) \hspace{1cm} \( \wedge I, 1, 6 \)
8. .
9. .
Assumptions & Subproofs

- know what an assumption is and that assumptions start subproofs
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- know that you can reason within a subproof (under an assumption)
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- know what an assumption is and that assumptions start subproofs
- know that subproofs can be nested or independent of each other
- know that you can reason within a subproof (under an assumption)
- examine some general rules for reasoning in a subproofs and some violations of those rules
You can use derivation rules to reason within subproofs, but there are certain restrictions. The basic rule is the following:
You can use derivation rules to reason within subproofs, but there are certain restrictions. The basic rule is the following:

**Deriving wffs from a subproof**

If $P$ is in a section of the proof $S_1$ that contains another subproof $S_2$, then $P$ can be used in $S_2$. 
Assumptions & Subproofs

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**Deriving wffs from a subproof**
If $P$ is in a section of the proof $S_1$ that contains another subproof $S_2$, then $P$ can be used in $S_2$.

**Deriving wffs from a subproof**
In other words, if $R$ is in a section of the proof $S_3$ that does not contain a subproof $S_4$, then $R$ cannot be used in $S_4$. 
The above rule is violated when using a proposition inside a subproof, you derive a proposition outside the subproof.
Example # 1: $Z$ is in the subproof and used to derive $Z \land R$, a proposition which is not in the subproof containing $Z$

<table>
<thead>
<tr>
<th>1</th>
<th>$R$</th>
<th>P</th>
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<tr>
<td>2</td>
<td>$Z$</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>$R \land Z$</td>
<td>$\land I, 1, 2$</td>
</tr>
<tr>
<td>4</td>
<td>$Z \land R$, NO!</td>
<td>$\land I, 1, 2$</td>
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Example # 2: $B$ is in the subproof and used to derive $B \land C$, a proposition which is not in the subproof containing $B$

1. $A$ (P)
2. $B$ (A)
3. $A \land B$ ($\land I$, 1, 2)
4. $C$ (A)
5. $B \land C$, NO! ($\land I$, 2, 4)
Reasoning with Subproofs

- know what an assumption is and general rules for reasoning in a subproofs and some violations of those rules
Reasoning with Subproofs

- know what an assumption is and general rules for reasoning in a subproofs and some violations of those rules
- but what can we do with subproofs?
Reasoning with Subproofs

- know what an assumption is and general rules for reasoning in a subproofs and some violations of those rules
- but what can we do with subproofs?
- specific derivation rules that allow you to reason from a subproof to a wff in the proof.
Conditional Introduction ($\rightarrow \vdash$)

Conditional introduction says that from a subproof that begins with the assumption $P$ and ends with $Q$, we can derive the conditional $P \rightarrow Q$. 
Conditional Introduction (→ l)

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\[
\begin{array}{c|cc}
 n & P & A \\
 \vdots & \vdots & \vdots \\
 (n+1) & Q & \rightarrow l, \ n-(n+1) \\
 (n+2) & P \rightarrow Q & \\
\end{array}
\]
Conditional introduction

Here is an example: $R \vdash Z \rightarrow R$
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<td>$\land I, 1, 2$</td>
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<td>$R$</td>
<td>$\land E, 3$</td>
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<td>$\rightarrow I, 2–4$</td>
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Conditional introduction

- The basic idea behind conditional introduction is that if assume a wff $P$ and can show that some wff follows from $P$, e.g. $Q$, then you can derive $P \rightarrow Q$. 

Let's consider an example in plain English:

1. Assume that a nuclear bomb is dropped on my house (Note: I am not saying that one has been dropped on my house).
2. Under this assumption, I can infer that my house will be destroyed.
3. Therefore, I can reason using conditional introduction to the complex proposition: if a nuclear bomb is dropped on my house, then it will be destroyed.
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  2. Under this assumption, I can infer that my house will be destroyed.
  3. Therefore, I can reason using **conditional introduction** to the complex proposition: $\text{if a nuclear bomb is dropped on my house, then it will be destroyed.}$
Conditional introduction
Conditional Elimination \((\rightarrow E)\)

From \(P \rightarrow Q\) and \(P\), we can derive \(Q\).
Conditional Elimination ($\rightarrow E$)

From $P \rightarrow Q$ and $P$, we can derive $Q$.

$P \rightarrow Q, P \vdash Q$
Conditional Elimination ($\rightarrow E$)

From $P \rightarrow Q$ and $P$, we can derive $Q$.

$P \rightarrow Q, P \vdash Q$

Conditional elimination allows for deriving a proposition $Q$ provided we have a conditional $P \rightarrow Q$ and the antecedent $P$ of that conditional.
Conditional Elimination \((\to E)\)

Here is an example: \(Z \to R, Z \land P \vdash R\)
Conditional Elimination ($\rightarrow E$)

Here is an example: $Z \rightarrow R, Z \land P \vdash R$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$Z \rightarrow R$</td>
<td>$Z \land P$</td>
<td>$Z$</td>
<td>$R$</td>
</tr>
<tr>
<td></td>
<td>P</td>
<td>P , R</td>
<td>$\land E , 2$</td>
<td>$\rightarrow E , 1 , 3$</td>
</tr>
</tbody>
</table>
Conditional Elimination ($\rightarrow E$)
Reiteration (R)
From P we can derive P.
Reiteration (R)

From $P$ we can derive $P$. 

$P \vdash P$
Reiteration (R)

From P we can derive P.

\[ P \vdash P \]

Reiteration allows for deriving a proposition \( P \) provided \( P \) already occurs in the proof.
Here are two examples.
Here are two examples.

Example # 1: \( Z \vdash Z \)
Reiteration

Here are two examples.

Example # 1: $Z \vdash Z$

<p>| | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>$Z$</td>
<td>$P, Z$</td>
</tr>
<tr>
<td>2</td>
<td>$Z$</td>
<td>$R, 1$</td>
</tr>
</tbody>
</table>
Example # 2: $R \vdash Z \rightarrow R$
Reiteration

Example # 2: \( R \leftarrow Z \rightarrow R \)

1. \( R \)  \( P, Z \rightarrow R \)
2. \( Z \)  \( A \)
3. \( R \)  \( R, 1 \)
4. \( Z \rightarrow R \)  \( \rightarrow I, 2–3 \)
Negation Introduction (¬/)

From a derivation of a proposition Q and its literal negation ¬Q within a subproof involving an assumption P, we can derive ¬P out of the subproof.
Negation Introduction ($\neg I$)

From a derivation of a proposition $Q$ and its literal negation $\neg Q$ within a subproof involving an assumption $P$, we can derive $\neg P$ out of the subproof.

\[
\begin{array}{c|cc}
 n & P & A \\
\vdots & \vdots & \\
(n+1) & Q & \\
(n+2) & \neg Q & \\
(n+3) & \neg(P) & \neg I, \ n\neg(n + 2)
\end{array}
\]
Negation Elimination (¬E)

From a derivation of a proposition $Q$ and its literal negation $\neg Q$ within a subproof involving an assumption $\neg(P)$, we can derive $P$ out of the subproof.

\[
\begin{array}{c|ccc}
\text{n} & \neg(P) & \text{A} \\
\vdots & \vdots & \\
(n + 1) & Q & \\
(n + 2) & \neg Q & \\
(n + 3) & P & \neg E, \ n\neg(n + 2)
\end{array}
\]
Negation Elimination

Example: $P \land \neg P \vdash Z \rightarrow R$
Negation Elimination

Example: $P \land \neg P \vdash Z \rightarrow R$

1. $P \land \neg P$  
   $P, Z \rightarrow R$
2. $\neg (Z \rightarrow R)$  
   A
3. $P$  
   $\land E, 1$
4. $\neg P$  
   $\land E, 1$
5. $Z \rightarrow R$  
   $\neg E, 2–4$
1. $\neg E$ and $\neg I$ are a species of reductio ad absurdum ("reduction to absurdity")
Negation Elimination and Introduction

1. \( \neg E \) and \( \neg I \) are a species of reductio ad absurdum ("reduction to absurdity")
2. sometimes these proofs are called proof by contradiction
1. \(\neg E\) and \(\neg I\) are a species of reductio ad absurdum ("reduction to absurdity")
2. sometimes these proofs are called **proof by contradiction**
3. both are a form of **indirect proof** as they prove that something is the case by (i) assuming the *opposite* proposition and (ii) showing that proposition to lead to contradiction (\(\neg Q, Q\)).
Negation Elimination and Introduction

1. \( \neg E \) and \( \neg I \) are a species of reductio ad absurdum ("reduction to absurdity")
2. sometimes these proofs are called proof by contradiction
3. both are a form of indirect proof as they prove that something is the case by (i) assuming the opposite proposition and (ii) showing that proposition to lead to contradiction (\( \neg Q, Q \)).
4. How might you use \( \neg E \) and \( \neg I \) in everyday life?
1. Premises: If God exists, then God is all-loving, all-knowing, and all-powerful. If God is all-loving, all-knowing, and all-powerful, then there should be no evil in the world. There is evil in the world.
1. Premises: If God exists, then God is all-loving, all-knowing, and all-powerful. If God is all-loving, all-knowing, and all-powerful, then there should be no evil in the world. There is evil in the world.

2. Suppose you wanted to accept the above and you want to prove that God does not exist.
1. Premises: If God exists, then God is all-loving, all-knowing, and all-powerful. If God is all-loving, all-knowing, and all-powerful, then there should be no evil in the world. There is evil in the world.

2. Suppose you wanted to accept the above and you want to prove that God does not exist.

3. Start by assuming the opposite of what you want to prove. That is, you would argue by saying Let’s assume that God does exist.
1. Premises: If God exists, then God is all-loving, all-knowing, and all-powerful. If God is all-loving, all-knowing, and all-powerful, then there should be no evil in the world. There is evil in the world.

2. Suppose you wanted to accept the above and you want to prove that God does not exist.

3. Start by assuming the opposite of what you want to prove. That is, you would argue by saying Let’s assume that God does exist.

4. Then you would try to show that under this assumption, a contradiction follows (there should be no evil in the world and there is evil in the world), and so ultimately conclude that God does not exist.
Time for some frequently asked questions!
Question
What should I assume?

You will always assume the opposite (the negation) of the wff you want.
1. If you want to derive \( P \) (it's your conclusion), then assume \( \neg P \).
2. If you want to derive \( P \land Q \), then assume \( \neg (P \land Q) \).
3. If you want to derive \( \neg Z \), then assume either \( Z \) or \( \neg \neg Z \).
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Negation Elimination and Introduction: FAQs

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Question

When is it a good time to use ¬E and ¬I?
Negation Elimination and Introduction: FAQs

Question
When is it a good time to use \( \neg E \) and \( \neg I \)?

1. when the premises are inconsistent.
Question

When is it a good time to use $\neg E$ and $\neg I$?

1. when the premises are inconsistent.
   - Derive $\phi$ and $\neg(\phi)$, then assume the opposite of the conclusion, reiterate $\phi$ and
     $\neg(\phi)$ into the subproof, then use either $\neg E$ and $\neg I$
Question

When is it a good time to use $\neg E$ and $\neg I$?

1. when the premises are inconsistent.
   - Derive $\phi$ and $\neg(\phi)$, then assume the opposite of the conclusion, reiterate $\phi$ and $\neg(\phi)$ into the subproof, then use either $\neg E$ and $\neg I$

2. when you can’t use any more elimination rules to simplify your proof or you don’t know what to do.
Negation Elimination and Introduction: FAQs

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When is it a good time to use $\neg E$ and $\neg I$?

1. when the premises are inconsistent.
   - Derive $\phi$ and $\neg (\phi)$, then assume the opposite of the conclusion, reiterate $\phi$ and $\neg (\phi)$ into the subproof, then use either $\neg E$ and $\neg I$

2. when you can’t use any more elimination rules to simplify your proof or you don’t know what to do.
   - Assume the opposite of the conclusion, then see if this assumption will allow you to derive $\phi$ and $\neg (\phi)$. If it does, then you can use either $\neg E$ and $\neg I$ to solve.
Question

When is it NOT a good time to use $\neg E$ or $\neg I$?
Question

When is it **NOT** a good time to use ¬E or ¬I?

1. when you don’t have a clear reason to do it and you can use elimination rules (e.g., ∧E, →E) to simplify your proof further
Question
When is it NOT a good time to use $\neg E$ or $\neg I$?

1. when you don’t have a clear reason to do it and you can use elimination rules (e.g., $\land E$, $\to E$) to simplify your proof further

2. Maybe: when your conclusion is a conditional. Here it might make more sense to try and use $\to I$
Question
When can I use these rules in daily life?

1. Practically: hard to say. You are showing a position to be absurd.
2. Argumentative situation: when you think your opponent's position entails an absurdity (you can assume their position to be true, draw out the contradiction, and then reason to the conclusion that their position is false).
3. Personal situation: use as a way of self-critique. Suppose I think that (i) character is determined by what you do on a regular basis, (ii) I don't give to charity (or at all). I might assume I'm charitable and reason to a contradiction.
Question

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Negation Elimination and Introduction: FAQs

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3. **personal situation:** use as a way of self-critique. Suppose I think that (i) character is determined by what you do on a regular basis, (ii) I don’t give to charity (or at all). I might assume I’m charitable and reason to a contradiction.
Can I see you use this rule one more time? Example please!
Negation Elimination and Introduction: FAQs
Disjunction Introduction
In the case of disjunction introduction ($\lor$), we reason as follows:

- $P1$: Proposition $P$ is the case.
- $C$: Therefore, $P$ or $Q$ is the case.
Disjunction Introduction

People tend not to reason using $\lor$ in everyday life. But, if they did, it would be perfectly acceptable.
People tend not to reason using $\lor$ in everyday life. But, if they did, it would be perfectly acceptable.

**Example**

1. P1: I will definitely take another philosophy class next semester.
People tend not to reason using $\lor$ in everyday life. But, if they did, it would be perfectly acceptable.

**Example**

1. P1: I will definitely take another philosophy class next semester.
2. C: Therefore, I will definitely take another philosophy class next semester OR a biology class.
People tend not to reason using $\lor$ in everyday life. But, if they did, it would be perfectly acceptable.

**Example**

1. P1: I will definitely take another philosophy class next semester.
2. C: Therefore, I will definitely take another philosophy class next semester OR a biology class.

**Example**

1. P1: John will look at the stars tonight.
People tend not to reason using $\lor$ in everyday life. But, if they did, it would be perfectly acceptable.

**Example**

1. P1: I will definitely take another philosophy class next semester.
2. C: Therefore, I will definitely take another philosophy class next semester OR a biology class.

**Example**

1. P1: John will look at the stars tonight.
2. C: Therefore, John will look at the stars tonight OR binge watch Netflix.
Disjunction introduction (as the name implies) is a derivation rule that introduces into the proof a disjunction (an “or” proposition, a wff where \( \vee \) is the main operator.)
Disjunction Introduction ($\lor I$)

From a wff $\phi$, $\phi \lor \psi$ or $\psi \lor \phi$ can be derived.

$\phi \vdash \phi \lor \psi$
$\phi \vdash \psi \lor \phi$
Disjunction Introduction

- Sometimes it isn’t obvious why $\lor I$ is a legitimate derivation rule
- Its acceptance can be seen through a truth-table analysis of the derivation $P \vdash P \lor Q$

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \vdash P \lor Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
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<tr>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

Notice that whenever $P$ is true, $P \lor Q$ is true. We can never go from a true premise to a false conclusion!
Double Negation (DN)

Where $P$ is any PL-wff, from $P$, we can derive $\neg\neg(P)$ and from $\neg\neg(P)$, we can derive $P$.

$P \vdash \neg\neg(P)$
Double Negation ($DN$)

Where $P$ is any PL-wff, from $P$, we can derive $\neg\neg(P)$ and from $\neg\neg(P)$, we can derive $P$.

$P \equiv \neg\neg(P)$

DN allows for replacing a single formula or single subformula with its doubly negated form or taking a doubly negated formula and replacing it with its unnegated form. For example,
Double Negation ($\text{DN}$)

1. $P \rightarrow R$ \hspace{1cm} P, $\neg \neg (P \rightarrow R)$
2. $\neg \neg (P \rightarrow R)$ \hspace{1cm} DN, 1
Double Negation ($DN$)

It is important to note that replacement rules can be applied to a single subformula. For example,

1. $P \lor \neg \neg (R \land S)$  \hspace{1cm} $P/P \lor (R \land S)$  \hspace{1cm} $P/P \lor (R \land S)$
2. $P \lor (R \land S)$  \hspace{1cm} DN, 1
Double Negation ($DN$)

But, be careful! DN must be applied to the whole of a formula or subformula and not to part of one subformula and part of another subformula:

1. $P \lor (R \land S)$ \hspace{1cm} P
2. $P \lor \neg(\neg R \land S)$, NO! \hspace{1cm} DN, 1
Disjunction Introduction

**Tricky Point**

One strange thing about $\lor$ is that you can reason from a wff $\phi$ to a disjunction $\phi \lor \psi$ and the wff $\psi$ can be any wff (even a contradiction!). This is because if $v(\phi) = T$, then $v(\phi \lor \psi) = T$. 

\[
\begin{align*}
\text{Example} & \\
P1 & : P \\
C & : P \lor \neg \neg (Q \lor S)
\end{align*}
\]
Disjunction Introduction

**Tricky Point**

One strange thing about $\lor$ is that you can reason from a wff $\phi$ to a disjunction $\phi \lor \psi$ and the wff $\psi$ can be any wff (even a contradiction!). This is because if $v(\phi) = T$, then $v(\phi \lor \psi) = T$.

**Example**

- P1: P
- C: $P \lor \neg\neg(Q \lor S)$

If $P$ is true, then $P \lor \neg\neg(Q \lor S)$ is true since a disjunction is true provided one of its disjuncts are true.
Disjunction Introduction

Let’s look at some examples of $\lor I$ in a proof. Let’s prove $R \vdash R \lor Q$
Let’s look at some examples of \( \vee I \) in a proof. Let’s prove \( R \vdash R \vee Q \)

\[
\begin{array}{ll}
1 & R & P \\
2 & R \vee Q & \vee I, 1 \\
\end{array}
\]

Notice that the justification for line (2) is line (1) and we cite \( \vee I \) to let everyone know what derivation rule we used.
Let’s look at some examples of $\lor I$ in a proof. Let’s prove $P \vdash (P \lor Q) \lor \neg R$.
Let’s look at some examples of $\lor I$ in a proof. Let’s prove $P \vdash (P \lor Q) \lor \neg R$.

1. $P$  \quad $P$  \
2. $P \lor Q$  \quad $\lor I$, 1  \
3. $(P \lor Q) \lor \neg R$  \quad $\lor I$, 2
Generally, there is no reason to use $\lor$ unless you **know you want a disjunction**. Using this rule to use it just makes the proof more complicated.

Suppose our proof is $(P \lor Q) \rightarrow R, P \vdash R$
Disjunction Introduction

Generally, there is no reason to use $\lor$ unless you know you want a disjunction. Using this rule to use it just makes the proof more complicated.

Suppose our proof is $(P \lor Q) \to R, P \vdash R$

1. $(P \lor Q) \to R \quad P$
2. $P \quad P$

Question: What rule should we use?
Prove: \((P \lor Q) \rightarrow R, P \vdash R\)

1. \((P \lor Q) \rightarrow R\) \hspace{1cm} P
2. \hspace{1cm} P

One idea is that if we had the disjunction \(P \lor Q\), then we could use one of our other rules (e.g. \(\rightarrow E: \phi \rightarrow \psi, \phi \vdash \psi\)) to derive \(R\).
Prove: \((P \lor Q) \rightarrow R, P \vdash R\)

1. \((P \lor Q) \rightarrow R\)  
   \(P\)
2. \(P\)  
   \(P\)
3. \(P \lor Q\)  
   \(\lor I, 2\)
Prove: \((P \lor Q) \rightarrow R, P \vdash R\). Now that we have \(P \lor Q\), we can use conditional elimination to derive our conclusion.

1. \((P \lor Q) \rightarrow R\)  
2. \(P\)  
3. \(P \lor Q\)  
4. \(R\)  

\(\rightarrow E, 1, 3\)
Disjunction Elimination
Disjunction elimination is perhaps the most complicated rule. Here is the basic idea:

- Start from a disjunction (Either P or Q).
- Assume P is the case and show that R follows from P.
- Now assume Q is the case and show that R follows from Q.
- Since you know either P or Q is the case, and you know that R follows from both P and Q, you can conclude that R is the case.
First, let me give you a visual example of this rule. Text of this visual example follows in case you forget what these crazy pictures mean.
Disjunction Elimination

- I am either going to stay home or I am going to party.
Disjunction Elimination

- I am either going to stay home or I am going to party.
- Will I have a good time tonight?

\[
\text{Well, assume I stay home, and I know that if I did this, I would have a good time.}
\]
\[
\text{Will I have a good time tonight?}
\]
\[
\text{If I stay home, I will, but I might decide to party instead. So we need to see what happens if I party!}
\]

\[
\text{Now assume that I party all night long, and I know that if I did this, I would have a good time.}
\]
\[
\text{Will I have a good time tonight?}
\]
\[
\text{If I party all night long, then I will have a good time.}
\]

RESULT:

- Since we know I’m either going to stay home or party, and we know that no matter what I do, I will have a good time, we can conclude I will have a good time.
Disjunction Elimination

- I am either going to stay home or I am going to party.
- **Will I have a good time tonight?**
- Well, assume I stay home, and I know that if I did this, I would have a good time.

If I stay home, I will, but I might decide to party instead. So we need to see what happens if I party!

- Now assume that I party all night long, and I know that if I did this, I would have a good time.
- **Will I have a good time tonight?**
- If I party all night long, then I will have a good time.

**RESULT:** Since we know I'm either going to stay home or party, and we know that no matter what I do, I will have a good time, we can conclude I will have a good time.
Disjunction Elimination

- I am either going to stay home or I am going to party.
- **Will I have a good time tonight?**
- Well, assume I stay home, and I know that if I did this, I would have a good time.
- **Will I have a good time tonight?** If I stay home, I will, but I might decide to party instead. So we need to see what happens if I party!

RESULT: Since we know I'm either going to stay home or party, and we know that no matter what I do, I will have a good time, we can conclude I will have a good time.
Disjunction Elimination

- I am either going to stay home or I am going to party.
- **Will I have a good time tonight?**
- Well, assume I stay home, and I know that if I did this, I would have a good time.
- **Will I have a good time tonight?** *If I stay home, I will, but I might decide to party instead. So we need to see what happens if I party!*
- Now assume that I party all night long, and I know that if I did this, I would have a good time.
Disjunction Elimination

- I am either going to stay home or I am going to party.
- **Will I have a good time tonight?**
- Well, assume I stay home, and I know that if I did this, I would have a good time.
- **Will I have a good time tonight?** If I stay home, I will, but I might decide to party instead. So we need to see what happens if I party!
- Now assume that I party all night long, and I know that if I did this, I would have a good time.
- **Will I have a good time tonight?** If I party all night long, then I will have a good time.
Disjunction Elimination

- I am either going to stay home or I am going to party.
- **Will I have a good time tonight?**
- Well, assume I stay home, and I know that if I did this, I would have a good time.
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- Now assume that I party all night long, and I know that if I did this, I would have a good time.
- **Will I have a good time tonight?** If I party all night long, then I will have a good time.
- **RESULT:** Since we know I’m either going to stay home or party, and we know that no matter what I do, I will have a good time, we can conclude I will have a good time.
NOTE!

Keep in mind that we are only saying “I will have a good time” follows from the premises. Not that “I will have a good time” is true. Logic is about drawing correct inferences from the premises! It might be the case that I have a terrible time since the disjunction is false: I might decide to neither stay home nor party.
Disjunction Elimination

Back to symbolic logic. The disjunction elimination rule (\(\lor E\)) states that from a disjunction \(\phi \lor \psi\) and two derivations of a wff \(\chi\) (where \(\chi\) can be any wff), the first where \(\phi\) is assumed and \(\chi\) is derived, the other where \(\psi\) is assumed and \(\chi\) is derived, we can derive \(\chi\).
# Disjunction Elimination

### Disjunction Elimination ($\lor E$)

<table>
<thead>
<tr>
<th></th>
<th>$P \lor Q$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$P$</td>
<td>$A$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td></td>
</tr>
<tr>
<td>$(n+1)$</td>
<td>$R$</td>
<td></td>
</tr>
<tr>
<td>$(i)$</td>
<td>$Q$</td>
<td>$A$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td></td>
</tr>
<tr>
<td>$(i+1)$</td>
<td>$R$</td>
<td></td>
</tr>
<tr>
<td>$(k)$</td>
<td>$R$</td>
<td>$\lor E, 1, n-(n+1), (i)-(i+1)$</td>
</tr>
</tbody>
</table>
Let’s look at an example where we try to prove the following:

\[ P \lor Q, P \rightarrow R, Q \rightarrow R \vdash R \]
Disjunction Elimination

Prove: $P \lor Q, P \rightarrow R, Q \rightarrow R \vdash R$. **First**, set up the proof.

1. $P \lor Q$  \quad P
2. $P \rightarrow R$  \quad P
3. $Q \rightarrow R$  \quad P
Prove: $P \lor Q, P \rightarrow R, Q \rightarrow R \vdash R$. **Next**, let’s assume $P$ as it is the left disjunct. In doing this, we want to derive a wff. Let’s try for $R$ since it is our conclusion!

1. $P \lor Q$  
2. $P \rightarrow R$  
3. $Q \rightarrow R$  
4. $P$  
5. $R \rightarrow E, 2, 4$
Disjunction Elimination

Prove: $P \lor Q, P \rightarrow R, Q \rightarrow R \vdash R$. **Now**, let’s assume $P$ as it is the right disjunct. In doing this, we want to derive the same wff as before, namely $R$.

1. $P \lor Q$  
P
2. $P \rightarrow R$  
P
3. $Q \rightarrow R$  
P
4. $P$  

   \hspace{1cm} \text{A}

5. $R$  

   \hspace{1cm} \rightarrow E, 2, 4

6. $Q$  

   \hspace{1cm} \text{A}

7. $R$  

   \hspace{1cm} \rightarrow E, 3, 6
Disjunction Elimination

Prove: $P \lor Q, P \rightarrow R, Q \rightarrow R \vdash R$. Finally, use $\lor E$ to derive $R$. Cite the rule ($\lor$), the disjunction (line 1), the first subproof (lines 4-5) and the second subproof (lines 6-7).

1. $P \lor Q$ P
2. $P \rightarrow R$ P
3. $Q \rightarrow R$ P
4. $P$ A
5. $R \rightarrow E, 2, 4$
6. $Q$ A
7. $R \rightarrow E, 3, 6$
8. $R \lor E, 1, 4-5, 6-7$
My general advice about $\lor E$ is to use it (1) as a last resort and (2) only when you have a disjunction already in the proof.
Biconditional Introduction
Biconditional Introduction ($\leftrightarrow /$)

Biconditional introduction, as the name implies, is a derivation rule that introduces into the proof a biconditional.
Suppose I can prove two propositions:

- If taxes go up, then there will be a recession.
- If there is a recession, then taxes have gone up.

Biconditional introduction is the procedure of proving the two propositions above and then derives the following biconditional:

- Taxes go up if and only if there will be a recession.
Biconditional Introduction ($\leftrightarrow$)

It does this in the following way:

1. First, assume $\phi$, then show that $\psi$ follows.
2. Second, assume $\psi$, then show that $\phi$ follows.
3. Finally, derive $\phi \leftrightarrow \psi$ from (1) and (2).
Biconditional Introduction

<table>
<thead>
<tr>
<th>n</th>
<th>$P$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<tr>
<td>(n + 1)</td>
<td>$Q$</td>
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<td>(i)</td>
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<tr>
<td>(i + 1)</td>
<td>$P$</td>
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</tr>
<tr>
<td>(k)</td>
<td>$P \leftrightarrow Q$</td>
<td>$\leftrightarrow l, n \rightarrow (n + 1), (i) \rightarrow (i + 1)$</td>
</tr>
</tbody>
</table>
Let’s consider an example where we prove: $P, Q \vdash P \leftrightarrow Q$
Biconditional Introduction

Prove: $P, Q \vdash P \leftrightarrow Q$. **First**, set up the proof.

\[
\begin{array}{c|cc}
1 & P & P \\
2 & Q & P \\
\end{array}
\]
Biconditional Introduction

Prove: $P, Q \vdash P \leftrightarrow Q$. **Next**, we need to assume $P$, then show $Q$ follows.

1. $P$   P
2. $Q$   P
3. $P$   A
4. $Q$   R, 2
Biconditional Introduction

Prove: \( P, Q \vdash P \leftrightarrow Q \). **Now**, we need to assume \( Q \), then show \( P \) follows.

\[
\begin{array}{c|c|c}
1 & P & P \\
2 & Q & P \\
3 & P & A \\
4 & Q & R, 2 \\
5 & Q & A \\
6 & P & R, 1 \\
\end{array}
\]
Biconditional Introduction ($\leftrightarrow \vdash$)

Prove: $P, Q \vdash P \leftrightarrow Q$. Finally, we can derive $P \leftrightarrow Q$ using

1. $P$  
2. $Q$  
3. $P$  

4. $Q$  
5. $P$  
6. $P$  

7. $P \leftrightarrow Q$  

$\leftrightarrow \vdash$, 3–4, 5–6
Biconditional elimination ($\leftrightarrow E$)
In the case of biconditional elimination ($\leftrightarrow E$), we reason as follows:
Biconditional Elimination ($\leftrightarrow E$)

In the case of biconditional elimination ($\leftrightarrow E$), we reason as follows:

- **P1**: P if and only if Q
- **P2**: P.
- **C**: Therefore, Q.

**OR**

- **P1**: P if and only if Q
- **P2**: Q.
- **C**: Therefore, P.
Biconditional Elimination ($\leftrightarrow E$)

Some examples:

Example 1. P1: John is a bachelor if and only if he is not married.
2. P2: John is not married.
3. C: Therefore, John is a bachelor.

Example 1. P1: I will go to heaven if and only if I am forgiven.
2. P2: I will go to heaven.
3. C: Therefore, I am forgiven.
Biconditional Elimination (↔ E)

Some examples:

Example

1. P1: John is a bachelor if and only if he is not married.
Some examples:

Example

1. P1: John is a bachelor if and only if he is not married.
2. P2: John is not married.
Biconditional Elimination ($\leftrightarrow E$)

Some examples:

**Example**

1. P1: John is a bachelor if and only if he is not married.
2. P2: John is not married.
3. C: Therefore, John is a bachelor.

**Example**

1. P1: I will go to heaven if and only if I am forgiven.
Some examples:

**Example**

1. P1: John is a bachelor if and only if he is not married.
2. P2: John is not married.
3. C: Therefore, John is a bachelor.

**Example**

1. P1: I will go to heaven if and only if I am forgiven.
2. P2: I will go to heaven.
Biconditional Elimination ($\leftrightarrow E$)

Some examples:

**Example**

1. P1: John is a bachelor if and only if he is not married.
2. P2: John is not married.
3. C: Therefore, John is a bachelor.

**Example**

1. P1: I will go to heaven if and only if I am forgiven.
2. P2: I will go to heaven.
3. C: Therefore, I am forgiven.
Biconditional Elimination ($\leftrightarrow E$)

From $P \leftrightarrow Q$ and $P$, we can derive $Q$. And, from $P \leftrightarrow Q$ and $Q$, we can derive $P$.

$P \leftrightarrow Q, P \vdash Q$
$P \leftrightarrow Q, Q \vdash P$
Biconditional Elimination ($\leftrightarrow E$)

1. $P \leftrightarrow Q$  
2. $P$  
3. $Q$  

$\leftrightarrow E$, 1, 2
Let’s solve a proof using $\leftrightarrow E$. Prove: $(P \land Q) \leftrightarrow S, P, Q \vdash S$. 
Let’s solve a proof using $\leftrightarrow E$. Prove: $(P \land Q) \leftrightarrow S$, $P$, $Q \vdash S$. **First,** set up the proof correctly.

1. $(P \land Q) \leftrightarrow S$  
2. $P$  
3. $Q$  

**Question:** Can we solve this proof in one step using $\leftrightarrow E$?
Prove: \((P \land Q) \leftrightarrow S, P, Q \vdash S\). **Second**, we see there is a biconditional at line 1. We need one side of the biconditional to reason to the other side. Let’s derive the left side \(P \land Q\).

\[
\begin{align*}
1 & \quad (P \land Q) \leftrightarrow S & \text{P} \\
2 & \quad P & \text{P} \\
3 & \quad Q & \text{P} \\
4 & \quad P \land Q & \land I, 2, 3
\end{align*}
\]
Prove: \((P \land Q) \leftrightarrow S, P, Q \vdash S\). **Now,** we have the left side of the biconditional. We can use \(\leftrightarrow E\) to derive the right side (which is our conclusion!)

1. \((P \land Q) \leftrightarrow S\)  
2. \(P\)  
3. \(Q\)  
4. \(P \land Q\) \(-I, 2, 3\)  
5. \(S\) \(-\leftrightarrow E, 1, 4\)
Proofs: Strategies
There are two main types of strategies: proof strategies and assumption strategies.

SP# 1 (E)  First, eliminate any conjunctions with $\land E$, disjunctions with $\lor E$, conditionals with $\rightarrow E$, and biconditionals with $\leftrightarrow E$. Then, if necessary, use any necessary introduction rules to reach the desired conclusion.

SP# 2 (B)  First, work backward from the conclusion using introduction rules (e.g. $\land I$, $\lor I$, $\rightarrow I$, $\leftrightarrow I$). Then, use SP# 1 (E).

Table 1: Proof Strategies
SA# 1 \((P, \neg Q)\) If the conclusion is an atomic wff (or a negated wff), assume the negation of the wff (or the non-negated form of the negated wff), derive a contradiction and then use \(\neg I\) or \(\neg E\).

SA# 2 \((\rightarrow )\) If the conclusion is a conditional, assume the antecedent, derive the consequent, and use \(\rightarrow I\).

SA# 3 \((\land )\) If the conclusion is a conjunction, you will need two steps. First, assume the negation of one of the conjuncts, derive a contradiction, and then use \(\neg I\) or \(\neg E\). Second, in a separate subproof, assume the negation of the other conjunct, derive a contradiction, and then use \(\neg I\) or \(\neg E\). From this point, a use of \(\land I\) will solve the proof.

SA#4 \((\lor )\) If the conclusion is a disjunction, assume the negation of the whole disjunction, derive a contradiction, and then use \(\neg I\) or \(\neg E\).
Consider \( \neg (\neg P \land \neg Q) \vdash P \lor Q \). The strategy associated with assumptions is SA\#3.

\[
\begin{array}{c|c|c}
1 & \neg(\neg P \land \neg Q) & P, P \lor Q \\
2 & \neg(P \lor Q) & A/P, \neg P \\
\vdots & \vdots & \\
\end{array}
\]
The subgoal at this point is to generate a proposition P and its literal negation \(\neg P\) in the subproof, but it is not clear how to do this. You cannot generate P and \(\neg P\) out of nothing so consider what propositions you do have and try to derive a proposition that is a literal negation of these.

\[
\begin{align*}
1 & \quad \neg(\neg P \land \neg Q) \quad P, P \lor Q \\
2 & \quad \neg(P \lor Q) \quad A/P, \neg P \\
\vdots & \quad \vdots \\
\# & \quad \neg P \land \neg Q \text{ or } P \lor Q \quad ?
\end{align*}
\]
We thus have two options:

Option # 1: derive \( \neg P \land \neg Q \) since \( \neg (\neg P \land \neg Q) \) is its literal negation

Option # 2: derive \( P \lor Q \) since \( \neg (P \lor Q) \) is its literal negation
Consider option # 2. If we were to try to derive $P \lor Q$, we need to make an assumption, and the strategic rule associated with deriving disjunctions SA#4 says to assume the negation, derive $P$ and $\neg P$, and then use $\neg E$ or $\neg I$. In the case of the above proof, the next step would be as follows:

\[
\begin{align*}
1 & \quad \neg (\neg P \land \neg Q) & P, P \lor Q \\
2 & \quad \neg (P \lor Q) & A/P, \neg P \\
3 & \quad \quad \quad \quad \neg (P \lor Q) & A/P, \neg P \\
\vdots & \quad \vdots & \vdots \\
\# & \quad \neg P \land \neg Q \text{ or } P \lor Q & ?
\end{align*}
\]
But this does not help since we still have no way to get $P$, $\neg P$ in the proof. So, consider option # 1. If we were to try and derive $\neg P \land \neg Q$, we would need to make an assumption, and the strategic rule associated with conjunctions SA# 3($\land$) says to assume the literal negation of each of the conjuncts in separate subproofs, derive $P$ and $\neg P$ in each, and then use $\neg I$ or $\neg E$. 
1  \( \neg(\neg P \land \neg Q) \quad P, P \lor Q \)

2  \( \neg(P \lor Q) \quad A/P, \neg P \)

3  \( P \quad A/P, \neg P \)

\vdots

n  \( Q \quad A/P, \neg P \)

\vdots
Now the proof can be more easily solved.
Proofs: Additional Derivation Rules (PD+)
The set of 10 intelim rules along with reiteration forms PD, a derivation system capable of proving any valid argument in PL. In other words, PD consists of all of the essential derivation rules we need. However, you may have noticed that the proofs for many straightforwardly valid arguments are overly difficult or time-consuming. For example, the proof of $P \lor Q, \neg Q \vdash P$ is overly complicated given that the argument is straightforwardly valid. In what follows, a number of additional derivation rules are added to PD to form PD+. These additional derivation rules serve to expedite the proof solving process.
Disjunctive Syllogism
Disjunctive Syllogism (DS)

Disjunctive syllogism (DS) is a derivation rule in propositional logic.

1. it is a derived rule in an intelim system
2. made use of because (i) it corresponds to how people reason naturally and (ii) simplifies certain proofs
Disjunctive Syllogism (DS)

From $\phi \lor \psi$ and $\neg(\psi)$, we can derive $\phi$.

From $\phi \lor \psi$ and $\neg(\phi)$, we can derive $\psi$.

$\phi \lor \psi, \neg(\psi) \vdash \phi$

$\phi \lor \psi, \neg(\phi) \vdash \psi$
Disjunctive Syllogism (DS): Simple Example

1. \( P \lor R \)  
2. \( \neg R \)  
3. \( P \)  

DS, 1, 2
Disjunctive Syllogism (DS): English Example

- P1: There is cake or ice cream. \((C \lor I)\)
- P2: It is not the case that there is ice cream. \((\neg I)\).
- C: Therefore, there is cake. \((C)\)
The rule is justified in two different ways:

1. since DS is derived, we can prove $\phi \lor \psi, \neg(\phi) \vdash \phi$ using a more basic set of derivation rules.

2. intuitive justification from cases
Disjunctive Syllogism (DS): Intuitive justification

- P1: There is cake or ice cream. 
  \((C \lor I)\).
- P2: It is not the case that there is ice cream. \((\neg I)\).
- C: Therefore, there is cake. \((C)\)

- P1: (1) cake, (2) ice cream, (3) cake and ice cream

- P1: (1) cake, (2) ice cream, (3) cake and ice cream
Disjunctive Syllogism (DS): Intuitive justification

• P1: There is cake or ice cream. 
  \((C \lor I)\).

• P2: It is not the case that there is ice cream. 
  \((\neg I)\).

• C: Therefore, there is cake. \((C)\)
Disjunctive Syllogism (DS): Intuitive justification

- P1: There is cake or ice cream. 
  \((C \lor I)\).
- P2: It is not the case that there is ice cream. 
  \((\neg I)\).
- C: Therefore, there is cake. \((C)\)

- P1: (1) cake, (2) ice cream, (3) cake and ice cream
- P2: (1) cake, (2) ice cream, (3) cake and ice cream
- C: Therefore, there is cake.
Disjunctive Syllogism (DS): Example

1. $P \lor \neg(M \rightarrow T)$  \quad P
2. $\neg\neg(M \rightarrow T)$  \quad P
3. $P$  \quad DS, 1, 2
Modus Tollens (MT)

From \( P \rightarrow Q \) and \( \neg Q \), we can derive \( \neg P \).

The general idea is that given a conditional \( P \rightarrow Q \) and the literal negation of the consequent \( \neg Q \), the negation of the antecedent \( \neg P \) can be derived.
Modus Tollens (MT)

1. $P \rightarrow (S \lor R)$
2. $\neg(S \lor R)$
3. $\neg P$

$\neg P$ by MT, 1, 2
Hypothetical Syllogism
Hypothetical Syllogism (HS)

From $P \rightarrow Q$ and $Q \rightarrow R$, we can derive $P \rightarrow R$.

The idea is that if you have two conditionals $P \rightarrow Q$ and $Q \rightarrow R$ where the consequent of one conditional $P \rightarrow Q$ is the antecedent of the other conditional $Q \rightarrow R$, then you can derive a third conditional $P \rightarrow R$. 
Hypothetical Syllogism (HS)

In HS, we are reasoning from two conditionals to a third conditional.

1. $\phi \rightarrow \psi$  
2. $\psi \rightarrow \chi$  
3. $\phi \rightarrow \chi$  

- notice that the consequent of one conditional is the antecedent of another conditional.

- notice that the conditional we reason to is the antecedent of one conditional and the consequent of another
Individuals use *HS* in everyday reasoning all the time. Here is an example:

- P1: If I go to the store, I’ll buy some cigarettes.
- P2: If I buy cigarettes, I’ll end up smoking.
- C: Therefore, if I go to the store, I’ll end up smoking.

Notice you have the same proposition in part of each conditional!
Hypothetical Syllogism (HS)

It is helpful to think of rules using visual examples. Take a look at the following premises. What would the conclusion be if you used HS?
Hypothetical Syllogism (HS)

It is helpful to think of rules using visual examples. Take a look at the following premises. What would the conclusion be if you used HS?

1. \( \Rightarrow \Rightarrow \rightarrow P \)
2. \( \Rightarrow \Rightarrow \rightarrow \equiv \rightarrow P \)
3. \( \Rightarrow \equiv \rightarrow \equiv 1,2 \text{HS} \)
Hypothetical Syllogism (HS)

**When should you think about using HS?**
You should think about using HS when you see you have more than one conditional. If you have more than one conditional, then check if the antecedent of a conditional matches the consequent of another conditional.

**When shouldn’t you use HS?**
You shouldn’t really think about using HS if you don’t see any conditionals in your proof.
Hypothetical Syllogism ($HS$)

Let’s look at some examples. Example: $P \rightarrow Q$, $Q \rightarrow Z \vdash P \rightarrow Z$

1. $P \rightarrow Q$  \hspace{1em} P
2. $Q \rightarrow Z$  \hspace{1em} P
Let’s look at some examples. Example: $P \rightarrow Q, Q \rightarrow Z \vdash P \rightarrow Z$

1. $P \rightarrow Q$  
P  
2. $Q \rightarrow Z$  
P  
3. $P \rightarrow Z$  
HS, 1, 2
Hypothetical Syllogism ($HS$)

Example: $((P \land S) \rightarrow \neg Q, \neg Q \rightarrow \neg Z) \vdash (P \land S) \rightarrow \neg Z$

1. $(P \land S) \rightarrow \neg Q$  P
2. $\neg Q \rightarrow \neg Z$  P
Hypothetical Syllogism (HS)

Example: \((P \land S) \rightarrow \neg Q, \neg Q \rightarrow \neg Z \vdash (P \land S) \rightarrow \neg Z\)

1. \((P \land S) \rightarrow \neg Q\) \hspace{1cm} P
2. \(\neg Q \rightarrow \neg Z\) \hspace{1cm} P
3. \((P \land S) \rightarrow \neg Z\) \hspace{1cm} HS, 1, 2
Is this a correct use of HS?

1. $\neg M \to Z$ P
2. $P \to \neg M$ P
3. $P \to Z$ HS, 1, 2
Hypothetical Syllogism (HS)

Is this a correct use of HS?

1. \( \neg M \rightarrow Z \)  
2. \( P \rightarrow \neg M \)  
3. \( P \rightarrow Z \)  

\( \text{HS, 1, 2} \)

Yes!
Is this a correct use of HS?

1. \( P \rightarrow M \) \hspace{1cm} P
2. \( R \rightarrow Z \) \hspace{1cm} P
3. \( P \rightarrow Z \) \hspace{1cm} HS, 1, 2

Incorrect!
Hypothetical Syllogism (HS)

Is this a correct use of HS?

1. $P \rightarrow M$  
2. $R \rightarrow Z$  
3. $P \rightarrow Z$  

HS, 1, 2

Incorrect!

No! This is not correct. There is no wff that is the antecedent of one conditional and the consequent of another.
YOU TRY: $P \rightarrow M$, $M \rightarrow Z$, $Z \rightarrow W \vdash P \rightarrow W$
Hypothetical Syllogism (HS)

YOU TRY: $P \to M$, $M \to Z$, $Z \to W \vdash P \to W$

1. $P \to M$  P
2. $M \to Z$  P
3. $Z \to W$  P
Hypothetical Syllogism (HS)

YOU TRY: \( P \rightarrow M, M \rightarrow Z, Z \rightarrow W \vdash P \rightarrow W \)

\begin{align*}
1 & \quad P \rightarrow M & P \\
2 & \quad M \rightarrow Z & P \\
3 & \quad Z \rightarrow W & P \\
4 & \quad P \rightarrow Z & \text{HS, 1, 2}
\end{align*}
Hypothetical Syllogism (HS)

YOU TRY: $P \rightarrow M$, $M \rightarrow Z$, $Z \rightarrow W$ ⊢ $P \rightarrow W$

1. $P \rightarrow M$  P
2. $M \rightarrow Z$  P
3. $Z \rightarrow W$  P
4. $P \rightarrow Z$  HS, 1, 2
5. $P \rightarrow W$  HS, 4, 3
Hypothetical Syllogism (HS)

YOU TRY: $M \rightarrow Z$, $A \rightarrow \neg M$, $(A \land B) \rightarrow M \vdash (A \land B) \rightarrow Z$
YOU TRY: \( M \rightarrow Z, A \rightarrow \neg M, (A \land B) \rightarrow M \vdash (A \land B) \rightarrow Z \)

1. \( M \rightarrow Z \)  
2. \( A \rightarrow \neg M \)  
3. \( (A \land B) \rightarrow M \)
Hypothetical Syllogism ($HS$)

YOU TRY: $M \rightarrow Z, A \rightarrow \neg M, (A \land B) \rightarrow M \vdash (A \land B) \rightarrow Z$

1. $M \rightarrow Z$  P
2. $A \rightarrow \neg M$  P
3. $(A \land B) \rightarrow M$  P
4. $(A \land B) \rightarrow Z$  HS, 1, 3

Note
Line 2 has no role in solving this proof!
Note about HS
It is good to know HS, but you don’t need to ever use HS.

- HS is a derived rule.
- So, you can do everything HS does with the rules you already have.
- You use HS because it makes solving proofs simpler!
Hypothetical Syllogism ($HS$)

Try to solve this without HS: $P \rightarrow Q$, $Q \rightarrow R \vdash P \rightarrow R$

1. $P \rightarrow Q$  P
2. $Q \rightarrow R$  P

3. $P \rightarrow R$  $\vdash P \rightarrow R$
Hypothetical Syllogism ($HS$)

Try to solve this without $HS$: $P \rightarrow Q$, $Q \rightarrow R \vdash P \rightarrow R$

1. $P \rightarrow Q$  \hspace{1cm} P
2. $Q \rightarrow R$  \hspace{1cm} P
3. $P$  \hspace{1.5cm} A/R
4. $Q$  \hspace{1.5cm} $\rightarrow E$, 1, 3
5. $R$  \hspace{1.5cm} $\rightarrow E$, 2, 4
6. $P \rightarrow R$  \hspace{1.5cm} $\rightarrow I$, 3–5
Proofs: Additional Derivation Rules (PD+), The Replacement Rules
All of the previous derivation rules (besides DN) have been *inference rules*, these are derivation rules that allow for deriving a wff of one form from a wff(s) of another form.

In addition to adding DS, MT, and HS to PD, there are also *replacement rules*. Replacement rules are derivation rules that allow for replacing a wff or sub-formula of one type for a wff of a different type.
De Morgans Laws ($DeM$)

From $\neg(P \lor Q)$, we can derive $\neg P \land \neg Q$, and vice versa. From $\neg(P \land Q)$, we can derive $\neg P \lor \neg Q$.

De Morgans Laws ($DeM$)

$\neg(P \lor Q) \iff \neg P \land \neg Q$

$\neg(P \land Q) \iff \neg P \lor \neg Q$

The idea is that we can interchange a **negated disjunction** with a conjunction whose conjuncts are negated. And, we can interchange a **negated conjunction** with a disjunction whose disjuncts are negated.
Here is De Morgan’s laws in English:

1. “neither P nor Q” entails “not-P and not-Q” and vice versa
2. “not both P and Q” entails “not-P or not-Q” and vice versa
De Morgans Laws ($DeM$)

Example

“I want *neither* bread nor a cupcake” entails “I *don’t* want bread and I *don’t* want a cupcake.” (and vice versa)
De Morgans Laws \((DeM)\)

Example

“\(I\) don’t want \textbf{both} bread \textbf{and} a cupcake” entails “\(I\) \textbf{don’t} want bread \textbf{or} \(I\) \textbf{don’t} want a cupcake.” (and vice versa)
In the case of DeM:

- you can interchange a negated disjunction \( \neg(P \lor Q) \) with a conjunction whose conjuncts are negated \( \neg P \land \neg Q \) (and vice versa)
- you can interchange a negated conjunction \( \neg(P \lor Q) \) with a disjunction \( \neg P \lor \neg Q \) whose disjuncts are negated (and vice versa).
De Morgans Laws (DeM)

**Example:** Here DeM is applied to the negated disjunction \( \neg(R \lor S) \) to derive a conjunction with two negated conjuncts.

\[
\begin{align*}
1 & \quad \neg (R \lor S) & \quad \text{P} \\
2 & \quad \neg R \land \neg S & \quad \text{DeM, 1}
\end{align*}
\]

Notice that we are only using one line/wff when we use DeM. We cite this line and DeM in our justification column.
Example: Here De Morgan's laws are applied to $\neg R \land \neg Q$ to derive $\neg (R \lor Q)$, i.e. turning a conjunction with negated conjuncts into a negated disjunction.

1. $P \rightarrow (R \lor Q)$  
2. $\neg R \land \neg Q$  
3. $\neg (R \lor Q)$  

$\text{DeM, 2}$
It is helpful to have a visual metaphor for some rules. Did you ever play leapfrog as a child?
De Morgan's Laws ($DeM$)

When you use DeM (in the typical direction), you can think of the negations leaping over the parentheses onto each of the parts.
De Morgans Laws (DeM)

This metaphor is helpful when you are using DeM on complicated wffs.

Remember
The negation leaps over the parentheses and applies to each of the subformulas.

1 \( \neg (\neg P \lor \neg (Q \lor S)) \quad P \)
De Morgans Laws (DeM)

This metaphor is helpful when you are using DeM on complicated wffs.

Remember
The negation leaps over the parentheses and applies to each of the subformulas.

1 \( \neg(\neg P \vee \neg(Q \vee S)) \quad P \)

2 \( \neg\neg P \land \neg\neg(Q \vee S) \quad DeM, 2 \)

Remember
Also, don’t forget to change the sign from disjunction to conjunction OR conjunction to disjunction.
De Morgan's Laws ($DeM$)

When should you use DeM?

You should think about using DeM whenever you have a negated disjunction $\neg(P \lor Q)$ or negated conjunction $\neg(P \land Q)$.

- If you have a negated disjunction $\neg(P \lor Q)$ and use DeM, you derive $\neg P \land \neg Q$. Now that you have a conjunction, you can use $\land E$ to simplify your proof!
- If you have a negated conjunction $\neg(P \land Q)$ and use DeM, you derive $\neg P \lor \neg Q$. Now that you have a disjunction, you can look at rules like $\lor E$ or $DS$. 
Example: Here is an example we looked at earlier.

\[
\begin{align*}
1 & \quad \neg (R \lor S) \quad P \\
2 & \quad \neg R \land \neg S \quad DeM, \ 1
\end{align*}
\]
Example: Here is an example we looked at earlier.

1 \neg (R \lor S) \quad P
2 \neg R \land \neg S \quad DeM, 1
3 \neg R \quad \land E, 1
4 \neg S \quad \land E, 1

Notice that once we use DeM at line 2, we now can use \land E to derive \neg R and \neg S
Example: Here is another example!

1. \( \neg (R \land S) \)  
   \( \text{P} \)
2. \( \neg \neg S \)  
   \( \text{P} \)
3. \( \neg R \lor \neg S \)  
   \( \text{DeM, 1} \)
De Morgan’s Laws (DeM)

Example: Here is another example!

1  \( \neg (R \land S) \)  P
2  \( \neg \neg S \)  P
3  \( \neg R \lor \neg S \)  DeM, 1
4  \( \neg R \)  DS, 2, 3

Notice that once we use DeM at line 3, we now can use DS to derive \( \neg R \).
De Morgan’s Laws is a replacement rule. We are replacing a wff of one type with one of another type. This means we can also apply it to subformula.

1 \( P \rightarrow \neg(P \lor Q) \) P
2 \( P \rightarrow (\neg P \land \neg Q) \) Dem, 1

Notice

Note that DeM was applied to the consequent of the conditional.
De Morgans Laws \((DeM)\)

**DeM is not necessary**

We don’t need DeM since we can solve everything without it, but it is such a powerful rule as it simplifies so many **really hard** proofs!

If you are looking for a challenge, solve all **four directions** of DeM:

\[
\neg (P \lor Q) \iff \neg P \land \neg Q \quad \text{AND} \quad \neg (P \land Q) \iff \neg P \lor \neg Q
\]

1. \(\neg (P \lor Q) \iff \neg P \land \neg Q\)
2. \(\neg P \land \neg Q \iff \neg (P \lor Q)\)
3. \(\neg (P \land Q) \iff \neg P \lor \neg Q\)
4. \(\neg P \lor \neg \neg Q \iff \neg (P \land Q)\)
De Morgans Laws ($DeM$)

**When should I look to use DeM?**

You should look to use DeM whenever you see a negated conjunction or a negated disjunction.
De Morgan's Laws \((DeM)\)

**When should I look to use DeM?**

You should look to use DeM whenever you see a negated conjunction or a negated disjunction.

Think of something you think is **gross**! Really think about how it makes you feel!
De Morgans Laws \((DeM)\)

Negated conjunctions \(\neg(P \land Q)\) and negated disjunctions \(\neg(P \lor Q)\) are gross wffs because:

- You can’t do anything with these wffs
- You can’t use almost any rules on them

Whenever you see these gross wffs, you should try to replace them \(DeM\)!
You Try: \( \neg(P \lor T) \vdash \neg P \land \neg T \)

1. \( \neg(P \lor T) \quad P \)
De Morgans Laws \((DeM)\)

You Try: \(\neg(P \lor T) \vdash \neg P \land \neg T\)

1 \(\neg(P \lor T)\) \(\quad\) P
2 \(\neg P \land \neg T\) \(\quad\) DeM, 1
De Morgans Laws (DeM)

But Don’t Forget
While there is a “preferred” way to use DeM, don’t forget that DeM doesn’t only work in one direction!

Example: \( \neg P, \neg T \vdash \neg (P \lor T) \)

<table>
<thead>
<tr>
<th></th>
<th>( \neg P )</th>
<th>( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \neg P )</td>
<td>( P )</td>
</tr>
<tr>
<td>2</td>
<td>( \neg T )</td>
<td>( P )</td>
</tr>
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De Morgans Laws ($DeM$)

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<p>| | | |</p>
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<tr>
<td>1</td>
<td>$\neg P$</td>
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<tr>
<td>2</td>
<td>$\neg T$</td>
<td>$P$</td>
</tr>
<tr>
<td>3</td>
<td>$\neg P \land \neg T$</td>
<td>$\land I, 1, 2$</td>
</tr>
</tbody>
</table>
De Morgans Laws \((DeM)\)

**But Don’t Forget**

While there is a “preferred” way to use DeM, don’t forget that DeM doesn’t only work in one direction!

Example: \(\neg P, \neg T \vdash \neg (P \lor T)\)

1. \(\neg P\) \hspace{1cm} P
2. \(\neg T\) \hspace{1cm} P
3. \(\neg P \land \neg T\) \hspace{1cm} \(\land I, 1, 2\)
4. \(\neg (P \lor T)\) \hspace{1cm} \(\text{DeM, 3}\)
De Morgan's Laws ($DeM$)

You Try: $\neg(\neg R \lor (M \lor S)) \vdash \neg S$

1 \hspace{1cm} $\neg(\neg R \lor (M \lor S))$ \hspace{1cm} P

Remember the leapfrog metaphor!
De Morgan's Laws ($DeM$)

You Try: $\neg(\neg R \lor (M \lor S)) \vdash \neg S$

1 $\neg(\neg R \lor (M \lor S))$ P
2 $\neg\neg R \land \neg(M \lor S)$ $DeM, 1$
De Morgans Laws \((DeM)\)

You Try: \(\neg(\neg R \lor (M \lor S)) \vdash \neg S\)

1 \(\neg(\neg R \lor (M \lor S))\) \(P\)
2 \(\neg \neg R \land \neg (M \lor S)\) \(DeM, 1\)
3 \(\neg (M \lor S)\) \(\land E, 2\)
De Morgan’s Laws ($DeM$)

You Try: $\neg(\neg R \vee (M \vee S)) \vdash \neg S$

1. $\neg(\neg R \vee (M \vee S))$ P
2. $\neg\neg R \wedge \neg (M \vee S)$ $DeM$, 1
3. $\neg (M \vee S)$ $\wedge E$, 2
4. $\neg M \wedge \neg S$ $DeM$, 3
De Morgans Laws (DeM)

You Try: \( \neg(\neg R \lor (M \lor S)) \vdash \neg S \)

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Justification</th>
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<tbody>
<tr>
<td>1</td>
<td>( \neg(\neg R \lor (M \lor S)) )</td>
<td>P</td>
</tr>
<tr>
<td>2</td>
<td>( \neg \neg R \land \neg (M \lor S) )</td>
<td>DeM, 1</td>
</tr>
<tr>
<td>3</td>
<td>( \neg (M \lor S) )</td>
<td>( \land )E, 2</td>
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<tr>
<td>4</td>
<td>( \neg M \land \neg S )</td>
<td>DeM, 3</td>
</tr>
<tr>
<td>5</td>
<td>( \neg S )</td>
<td>( \land )E, 4</td>
</tr>
</tbody>
</table>
Let’s practice

Let’s conclude by looking at some proofs that may involve HS and DeM but also the other rules.
Let's practice

1. \( \neg (T \lor S), \neg (M \lor \neg R) \vdash \neg S \land R \)
2. \( P \rightarrow \neg (Q \lor S), (\neg Q \land \neg S) \rightarrow T \vdash P \rightarrow T \)
3. \( \neg (P \lor \neg (Q \rightarrow Z)), Z \rightarrow M \vdash Q \rightarrow M \)
4. \( \neg (P \land \neg (Q \rightarrow R)), P, R \rightarrow M \vdash Q \rightarrow M \)
5. \( \vdash P \lor \neg P \)
6. \( \vdash P \lor \neg P \) without DeM
Let's practice
Implication
Implication (IMP)

From $P \rightarrow Q$, we can derive $\neg P \lor Q$. From $\neg P \lor Q$, we can derive $P \rightarrow Q$. $P \rightarrow Q \dashv \vdash \neg P \lor Q$

In the case of IMP, you can interchange a negated conditional $P \rightarrow Q$ with a disjunction $\neg P \lor Q$. 
Implication (IMP)

Remembering that replacement rules can be applied to single sub-formula, notice how IMP is applied to the subformula $P \rightarrow R$ in $\neg(P \rightarrow R)$ in the following example:

1. $\neg(P \rightarrow R)$  
2. $\neg(\neg P \lor R)$  
3. $\neg\neg P \land \neg R$  
4. $\neg\neg P$  
5. $P$

1. P
2. IMP, 1
3. DeM, 2
4. \&E, 3
5. DN, 4
Proofs: Revised Strategic Rules
Strategic Proof Rules

In enhancing our proof system from $PD$ to $PD+$, we also want to enhance the strategies with which we solve proofs.

1. SP# 1 (E+)
   First, eliminate any conjunctions with $\land E$, disjunctions with $\lor E$ or $\lor I$, conditionals with $\rightarrow E$ or $\rightarrow I$, and biconditionals with $\leftrightarrow E$.
   Then, if necessary, use any necessary introduction rules to reach the desired conclusion.

2. SP# 2 (B)
   First, work backward from the conclusion using introduction rules (e.g. $\land I$, $\lor I$, $\rightarrow I$, $\leftrightarrow I$). Then, use SP# 1 (E).

3. SP# 3 (EQ+)
   Use DeM on any negated disjunctions or negated conjunctions, and then use SP# 1 (E). Use IMP on negated conditionals, then use DeM, and then use SP# 1 (E).
In enhancing our proof system from \textit{PD} to \textit{PD+}, we also want to enhance the strategies with which we solve proofs.

1. \textbf{SP\# 1 (E+)} First, eliminate any conjunctions with $\land E$, disjunctions with $DS$ or $\lor E$, conditionals with $\rightarrow E$ or $MT$, and biconditionals with $\leftrightarrow E$. Then, if necessary, use any necessary introduction rules to reach the desired conclusion.

2. \textbf{SP\# 2 (B)} First, work backward from the conclusion using introduction rules (e.g. $\land I$, $\lor I$, $\rightarrow I$, $\leftrightarrow I$). Then, use SP\# 1(E).

3. \textbf{SP\# 3 (EQ+)} Use DeM on any negated disjunctions or negated conjunctions, and then use SP\# 1(E). Use IMP on negated conditionals, then use \textit{DeM}, and then use SP\# 1(E).
Five Tips for Propositional Logic Proofs
Five Tips: Tip 1

- Know the different proposition types.
Five Tips: Tip 1

- Know the different proposition types.
- If $P \land Q$ is a conjunction, $\neg(P \lor Q)$ is a negated disjunction, $P \rightarrow Q$ is a conditional, and so on.
Five Tips: Tip 1

- Know the different proposition types.
- If $P \land Q$ is a conjunction, $\neg(P \lor Q)$ is a negated disjunction, $P \rightarrow Q$ is a conditional, and so on.
- Knowing the proposition type gives you information about what derivation rules you can apply: $P \land Q$ is a conjunction so look at derivation rules that can operate on conjunctions, $\neg(P \lor Q)$ is a negated disjunction so look at DeM, etc.
Five Tips: Tip 2

- Understand the distinction between **Introduction** and **Elimination** rules.
Five Tips: Tip 2

- Understand the distinction between **Introduction** and **Elimination** rules.
- Elim rule - reasoning from a wff of a particular type. $\wedge E$ is reasoning FROM a conjunction. So, if you are using conjunction elimination, you will be reasoning FROM a conjunction. That is, **from** $P \wedge Q$ **to** a wff.
Five Tips: Tip 2

- Understand the distinction between **Introduction** and **Elimination** rules.
- Elim rule - reasoning from a wff of a particular type. $\land E$ is reasoning FROM a conjunction. So, if you are using conjunction elimination, you will be reasoning FROM a conjunction. That is, **from** $P \land Q$ to a wff.
- Intro rule - reasoning to a wff of a particular type. $\land I$ is reasoning TO a conjunction. That is, **from** a set of wffs TO $P \land Q$. 
Five Tips: Tip 3

- Apply elimination rules to simplify ("break down") wffs in a proof.
Five Tips: Tip 3

- Apply elimination rules to simplify ("break down") wffs in a proof.
- \((P \land Q) \land R\) - conjunction, apply \(\land E\) to simplify this wff to \(P \land Q\), and \(R\), then you can use \(\land E\) to simplify this further to \(P\) and \(Q\).
Five Tips: Tip 3

- Apply elimination rules to simplify ("break down") wffs in a proof.
- \((P \land Q) \land R\) - conjunction, apply \(\land E\) to simplify this wff to \(P \land Q\), and \(R\), then you can use \(\land E\) to simplify this further to \(P\) and \(Q\).
- \(P \to Q\). Look to see if you can use \(\to E\). To use this rule you will want to see if there is \(P\) on its own line. If there is, you can use \(\to E\) on \(P \to Q\), \(P\) to reason to \(Q\).
Five Tips: Tip 4

• Think about the introduction rules you would have to apply to reason to the conclusion.
Five Tips: Tip 4

- Think about the introduction rules you would have to apply to reason to the conclusion.
- Conclusion: $P \lor Q$. This is a disjunction. Consider disjunction introduction ($\lor I$). If you had $P$ or $Q$, you could reason to $P \lor Q$. 

Five Tips: Tip 4

• Think about the introduction rules you would have to apply to reason to the conclusion.

• Conclusion: $P \lor Q$. This is a disjunction. Consider disjunction introduction ($\lor I$). If you had $P$ or $Q$, you could reason to $P \lor Q$.

• Conclusion: $P \to Q$. This is a conditional. Consider conditional introduction ($\to I$). Rule says Assume $P$, now if you can derive $Q$ in the subproof, you can use $\to I$ to derive $P \to Q$. 
If all else fails, try assuming the literal negation of your conclusion and deriving a contradiction within the subproof, then using $\neg E$ or $\neg I$. 

$\neg (P \lor P) / \neg W$. You are totally stumped.

Try assuming $\neg W$ (the negation of your conclusion), then deriving $P$ and $\neg P$ within the subproof. If you do this, then you can use $\neg E$ to derive $W$. 
Five Tips: Tip 5

- If all else fails, try assuming the literal negation of your conclusion and deriving a contradiction within the subproof, then using \(\neg E\) or \(\neg I\).
- \(\neg(P \lor P) \vdash W\). You are totally stumped.
Five Tips: Tip 5

- If all else fails, try assuming the literal negation of your conclusion and deriving a contradiction within the subproof, then using $\neg E$ or $\neg I$.

- $\neg (P \lor P) \vdash W$. You are totally stumped.

- Try assuming $\neg W$ (the negation of your conclusion), then deriving $P$ and $\neg P$ within the subproof. If you do this, then you can use $\neg E$ to derive $W$. 